Chapter 4

Sets

Philosophers have not found it easy to sort out sets . . .

D. M. Armstrong,

It is useful to have a way of describing a collection of "things" and the mathematical name for such a collection is a *set*. So the collection of colours {Red,Blue, Green } is a set we might call A and write as A={Red, Blue, Green } . Other examples are

1. $\{1, 3, 7, 14\}$
2. $\{1, 2, 3, 5, 7, 11 \ldots\}$ the set of all prime numbers.
3. { Matthew, Mark, Luke, John}
4. {k : k is an integer and k is divisible by 4} here the contents are defined by a rule.
5. { All songs available on iTunes} again the contents are defined by a rule.

We do not care about the order of the elements of a set so {1, 2, 3} *is the same as* {3,2,1}.

Of course we may want to do things with sets and there is a whole mathematical language attached as you might expect. For example you will often see the statement a *belongs* to the set A written as a $\in$ A. The symbol $\notin$ is, of course, the converse i.e. *does not belong to.*

So

- Mark $\in$ {Matthew, Mark, Luke, John}
- Abergail $\notin$ {Matthew, Mark, Luke, John} .

- $7 \in \{1,2,3,4,5,6,7\}$

There are some sets that have special symbols because they are used a lot. Examples are

1. The set with nothing in it, called the *empty set* is written as $\emptyset$.
2. $\mathbb{N} = \{1, 2, 3, \ldots\}$ the set of natural numbers.
3. $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ the integers.
4. $\mathbb{Q}$ = the set of fractions.
5. $\mathbb{R}$ = the set of real numbers.
6. The set that contains everything is called the *universal* set written S, U or $\emptyset$.

Finally we will write $\bar{A}$ when we mean the set of things which are not in A.

Subsets

It is probably obvious that some set are "bigger" than others, for example $\{A,B,C,D,E\}$ and $\{B,C,D\}$. We formalize this idea by defining *subsets*.

If the set B contains all the elements in the set A together with some others then we write $A \subset B$. We say that A is a *subset* of B. So
$\{$Matthew, Mark, Luke, John$\} \subset \{$Matthew, Mark, Luke, John, Thomas $\}$

We can of course write this the other way around, so $A \subset B$ is the same as $B \supset A$.

1. Formally for $A \subset B$ we say if $a \in A$ then $a \in B$ or

$$a \in A \Rightarrow a \in B$$

2. If B is a subset but might possibly be the same as A then we use $A \subseteq B$.
3. We will use $A = B$ to mean A contains *exactly the same* things as B. Note that if $A \subseteq B$ and $B \subseteq A$ then $A = B$.

   In our logical symbolism we have

$$(A \subseteq B) \wedge (B \subseteq A) \Rightarrow A = B.$$

The *power set* of A, written, $P(A)$, or 2^A, is the set of all subsets of A. So if $A = \{$ Matthew, Mark, Luke $\}$ then P(A) is the set with eight elements
$\{$ Matthew, Mark, Luke $\}$
$\{$ Matthew, Mark $\}$
$\{$ Matthew, Luke $\}$
$\{$ Mark, Luke $\}$
$\{$ Matthew $\}$
$\{$ Mark $\}$
$\{$ Luke $\}$
$\emptyset$
The number of elements in a set A is called the *cardinality* of A and written $\|A\|$. So if $A = \{$ Matthew, Mark, Luke, John$\}$ then $\|A\|$=4.

Venn Diagrams and Manipulating Sets

We intend to manipulate sets and it helps to introduce Venn diagrams to illustrate what we are up to. We can think of the universal set S as a rectangle and a set, say A as the interior of the circle drawn in S, see figure 4.1 The speckled area is

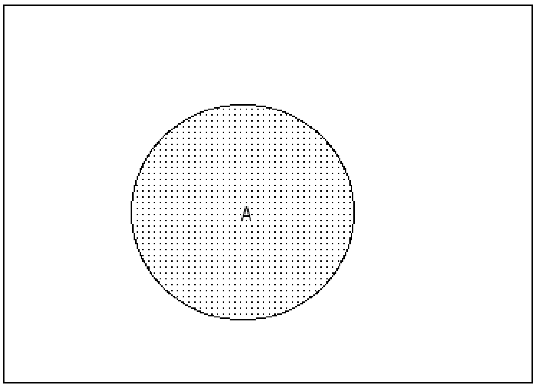

Figure 4.1: Venn diagram of set A and universal set S

A while the remainder of the area of the rectangle is $\bar{A}$. We see immediately that A together with $\bar{A}$ make up S

Intersection

We can write the set of items that belong to both the set A and the set B as $A \cap B$. Formally $(x \in A) \wedge (x \in B) \Rightarrow (x \in A \cap B)$.

We call this the *intersection* of A and B or, less formally, A *and* B. In terms of the Venn diagram in figure 4.2 the two circles represent A and B while the overlap (in black) is the intersection. As examples

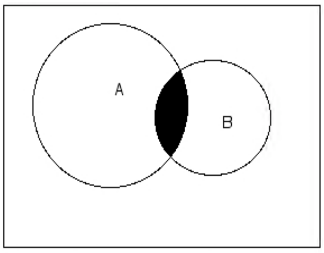

Figure 4.2: Venn diagram of $A \cap B$

1. $\{1,2,3,4\} \cap \{ 3,4,5,6,7\} =\{ 3,4\}$. Notice $3 \in \{ 3,4\}$ while $1 \notin \{ 3,4\}$.
2. $\{1,2,3,4\} \cap \{ 13,14,15,16,27\} =\emptyset$.
3. {Abergail, Ann, Blodwin, Bronwin, Clair,}$\cap$ { Abergail, Bronwin, Gareth, Ian} = {Abergail, Bronwin, }.
4. In figure 4.2 we see $A \cap \bar{A} = \emptyset$ so A and $\bar{A}$ have nothing in common.
5. $A \cap B \subset B$ and $A \cap B \subset A$

Union:

We can write the set of items that belong to the set A or the set B or to both as A $\cup$ B. Formally $(x \in A) \vee (x \in B) \Rightarrow x \in (A \cup B)$.

We call this the *union* of A and B or, less formally, A *or* B. The corresponding diagram is 4.3 Here the speckled area represents $A \cup B$

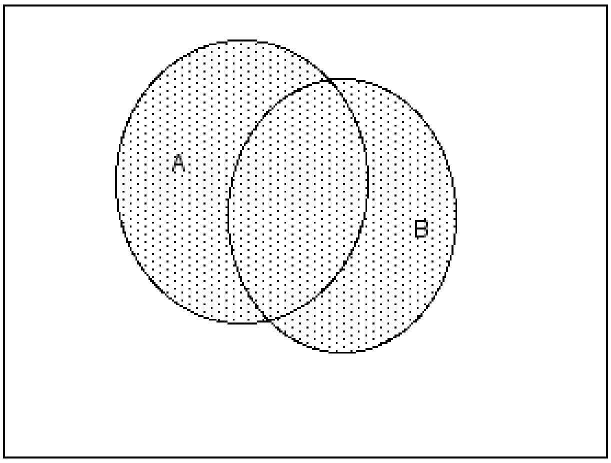

Figure 4.3: Venn diagram of set $A \cup B$ (speckled) and universal set S

As examples we have

1. $\{1,2,3,4\} \bigcup \{3,4,5,6,7\} = \{1,2,3,4,5,6,7\}$
2. $\{$ Blue,Green$\} \bigcup \{$ Red,Green$\} = \{$ Red,Blue , Green$\}$
3. In figure 4.2 we see $A \cup \bar{A} = S$ so A and $\bar{A}$ together make up S.
4. If $A \subset B$ then $A \cup B \subset B$

We can now use our basic definitions to get some results.

1. A= $\bar{\bar{A}}$ The set $\bar{A}$ consists of all the elements of S (the universal set) which do not belong to A. So $\bar{\bar{A}}$ is the set of elements that do not belong to $\bar{A}$, or the elements of S which do not belong to $\bar{A}$. That is the elements that belong to A.

   Or suppose $a \in \bar{\bar{A}} \Rightarrow a \notin \bar{A} \Rightarrow a \in A$

2. $\overline{(A \cap B)} = \bar{A} \cup \bar{B}$
   We have $a \in \overline{(A \cap B)} \Rightarrow a \notin (A \cap B) \Rightarrow (a \notin A) \vee (a \notin B) \Rightarrow (a \in \bar{A}) \vee (a \in \bar{B}) \Rightarrow a \in \bar{A} \cup \bar{B}$

There is a table of useful results in table 4.1. Notice each rule in the left column has a dual rule in the right. This dual has the $\cup$ symbol replace by $\cap$

$A \cup A = A$	$A \cap A = A$
$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
$A \cup B = B \cup A$	$A \cap B = B \cap A$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$A \cup \emptyset = A$	$A \cap S = A$
$A \cup S = S$	$A \cap \emptyset = \emptyset$
$A \cup \bar{A} = S$	$A \cap \bar{A} = \emptyset$
$\overline{(A \cup B)} = \bar{A} \cap \bar{B}$	$\overline{(A \cap B)} = \bar{A} \cup \bar{B}$

Table 4.1: Rules for set operations

Cartesian Product

Suppose we have two sets A and B. We define the Cartesian Product $P = A \times B$ to be the set of ordered pairs (a, b) where $a \in A$ and $b \in B$. Or

$$P = \{(a, b) : (a \in A) \wedge (b \in B)\}.$$

The pair (a, b) is ordered in the sense that the first term (a) comes from the set A in $A \times B$. The obvious example and hence the name comes from the geometry of the plane. We usually write (x, y) to denote the coordinates of a point on the plane. This is an ordered pair! If we take real values x and y with $x \in \mathbb{R}$ and $y \in \mathbb{R}$ then the Cartesian product is $\mathbb{R} \times \mathbb{R}$

1. Suppose $A = \{a, b\}$ and $B = \{1, 2\}$ then $A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$.

2. We can extend to 3 or more sets so $A \times B \times C$ is the set of ordered triples (a, b, c).

4.0.6 Relations and functions

Given two sets A and B and the product $A \times B$ we define a *relation* between A and B as a subset R of $A \times B$. We say that $a \in A$ and $b \in B$ are related if $(a, b) \in R$, more commonly written aRb. This is a quite obscure definition unless we look at the rule giving the subset.

Take the simple example of $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{1, 2, 3, 4, 5, 6\}$ then $A \times B$ is the array of pairs below - a set of 36 pairs.

$$\left\{ \begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array} \right\}$$

A *relation* R is the subset $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$ or the set $\{(i, j) : i = j\}$. Other example are

1. $R = \{(i, j) : i + j = 8\} = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$
2. $R = \{(i, j) : i = 2j\} = \{(2, 1), (4, 2), (6, 3)\}$
3. $R = \{i < j\} = \left\{ \begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ & & (2,3) & (2,4) & (2,5) & (2,6) \\ & & & (3,4) & (3,5) & (3,6) \\ & & & & (4,5) & (4,6) \\ & & & & & (5,6) \end{array} \right\}$

As you can see we can think of the relation R as a rule connecting elements of A to elements of B. The relation aRb between 2 sets A and B can be represented as in figure 4.4

For example

1. if A={ one, two, three, four, five} and $B = \{1, 2, 3, 4, 5\}$ we can define R as the set of pairs {(word, number of letters)} eg. {(one, 3), (two, 3), (three,5), . . .}
2. If A={ 2,4,8,16,32} and B={1,2,3,4,5} then we might define R as the set { (2,1),(4,2),(8,3),(16,4),(32,5)}

1. The *domain* of relation $\{(x, y)\}$ is the set of all the first numbers of the ordered pairs. In other words, the domain is all of the x-values.
2. The *range* of relation $\{(x, y)\}$ is the set of the second numbers in each pair, or the y-values.

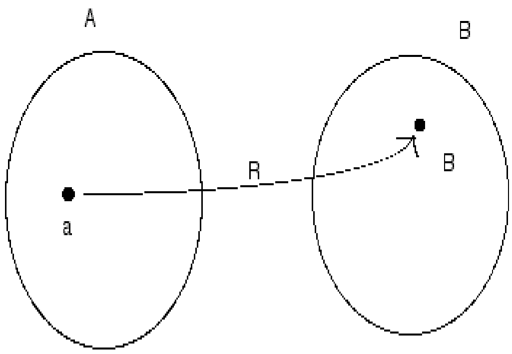

Figure 4.4: The relation R between 2 sets A and B

There are all kinds of names for special types of relations. Some of them are

1. *reflexive:* for all $x \in X$ it follows that xRx. For example, "greater than or equal to" is a reflexive relation but "greater than" is not.

2. *symmetric:* for all x and y in X it follows that if xRy then yRx. "Is a blood relative of" is a symmetric relation, because x is a blood relative of y if and only if y is a blood relative of x.

3. *antisymmetric:* for all x and y in X it follows that if xRy and yRx then $x = y$. "Greater than or equal to" is an antisymmetric relation, because if $x \geq y$ and $y \geq x$, then $x = y$.

4. *asymmetric:* for all x and y in X it follows that if xRy then not yRx. "Greater than" is an asymmetric relation, because $\mathtt{if} x > y$ then $y \not> x$.

5. *transitive:* for all x, y and z in X it follows that if xRy and yRz then xRz. "Is an ancestor of" is a transitive relation, because if x is an ancestor of y and y is an ancestor of z, then x is an ancestor of z.

6. *Euclidean:* for all x, y and z in X it follows that if xRy and xRz, then yRz.

7. A relation which is reflexive, symmetric and transitive is called an *equivalence relation.*

You can now speculate as the name "Relational Database".

exercises

1. If $A - B$ is the set of elements x that satisfy $x \in A$ and $x \notin B$ draw a Venn diagram for $A - B$

2. Prove that for sets A, B and C

   (a) If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$
   (b) If $A \subseteq B$ and $B \subset C$ then $A \subset C$
   (c) If $A \subset B$ and $B \subseteq C$ then $A \subset C$
   (d) If $A \subset B$ and $B \subset C$ then $A \subset C$

3. Recall that $\mathbb{Z} = \{0, 1, 2, 3, 4, \ldots\}$ and we define the following sets

   (a) $A = \{x \in \mathbb{Z} :$ for some integer $y > 0, x = 2y\}$
   (b) $B = \{x \in \mathbb{Z} :$ for some integer $y > 0, x = 2y - 1\}$
   (c) $A = \{x \in \mathbb{Z} :$ for some integer $x < 10\}$

   Describe $\bar{A}, (A \bar{\cup} B), \bar{C}, A - \bar{C},$ and$C - (A \cup B)$

4. Show that for all sets A, B and C

$$(A \cap B) \cup C = A \cap (B \cup C)$$

   iff $C \subseteq A$

5. What is the cardinalty of $\{\{1, 2\}, \{3\}, 1\}$.

6. Give the domain and the range of each of the following relations. Draw the graph in each case.

(a) $\{(x,y) \in \mathbb{R} \times \mathbb{R}\} \mid x^2 + 4y^2 = 1\}$

(b) $\{(x,y) \in \mathbb{R} \times \mathbb{R}\} \mid x^2 = y^2\}$

(c) $\{(x,y) \in \mathbb{R} \times \mathbb{R}\} \mid 0 \leq y, y \leq x$ and $x + 1y \leq 1\}$

7. Define the relation $\rhd$ between the ordered pairs $\{(x,y)$ and (u,v) where $x,y,v,v \in \mathbb{Z}\}$ where $(x,y) \rhd (u,v)$ means $xv = yu$. Show that $\rhd$ is an equivalence relation.